The Fractional-Order Goodwin Accelerator Model

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Abstract. The accelerator model proposed by Goodwin in 1951 is one of the pioneering nonlinear mathematical models of the business cycle. It has been studied in three different mathematical formulations, namely as a first-order delay differential equation, as a second-order ordinary differential equation and as a dynamical system of two first-order ordinary differential equations. All these formulations exhibit chaotic behavior. In this article, we analyze a fractional-order dynamical system of a specific form of the generalized dynamical system originating from the Goodwin accelerator model. We examine the steady-state stability of the commensurate as well as the incommensurate nonperturbed system. Subsequently, a numerical analysis of both the perturbed and the nonperturbed fractional-order system is conducted. Our main finding is that the incorporation of memory (or expectations) in the model can lead to local asymptotic stability of its equilibria and to less chaotic behavior. This can prove beneficial in modeling economic phenomena which are heavily dependent upon their past states.

Keywords: Fractional-Order Dynamical Systems, Accelerator-Multiplier Models, Economic Modeling

JEL classification: C61, *C62*, *E32*

1 Introduction

Mathematical methods have permeated mainstream economics since the Second World War. One of the very first areas of economic research to have been affected by the implementation of mathematics in economics is the study of business cycles. Frisch (1933) proposes one of the pioneering mathematical models of the business cycle, which is based upon the idea of the accelerator and the multiplier. It is formulated as a linear model. On the one hand, this rather simple mathematical structure makes it analytically solvable, but on the other hand it negatively impacts its ability to represent economic phenomena realistically[1]. As it turns out, linear models (albeit a powerful tool in mathematical modeling) generally fail to capture most of complexity of the real world. Goodwin (1951) tries to remedy this drawback of the previously-considered accelerator-multiplier models by deriving a model of his own[2]. Not only is this model nonlinear, which brings about some very interesting and chaotic behavior, but Goodwin also takes into account disinvestment and a time lag between decisions about investment and the corresponding outlays. These alterations make the model more realistic than its predecessors, though at the cost of greater mathematical complexity.

Despite all of the aforementioned efforts to make business-cycle modeling more realistic, one major aspect of economic phenomena is still largely overlooked when integer-order calculus is utilized. Economic systems possess a “memory” - i.e., they depend not only upon their current state (and its change in time), but they are highly influenced by their past states as well. This is the very reason why fractional-order calculus may prove beneficial when modeling real-world natural, technical as well as socioeconomic processes.

The main objective of this article is, therefore, to analyze the steady-state stability of the fractional-order Goodwin dynamical system of a specific form introduced in the text. In Section 2 the most relevant literature is discussed. In Section 3 methodology of which we make use throughout the article is briefly explained. We present the main results of our qualitative as well as numerical analysis in Section 4 and 5. Conclusions are drawn from the results and prospects for further research are considered in Section 6.

2 Literature Review

Goodwin (1951) presents a series of nonlinear dynamical business-cycle models with increasing mathematical complexity, which he hopes can better depict real-world economic processes than linear models proposed earlier. He eventually arrives at a first-order nonlinear delay differential equation (DDE; see Eqn. (4.1)). He then eliminates the delays by means of the Taylor series expansion. This gives rise to at least three different mathematical representations (all of which are laid out in Section 4) of the model. Goodwin (1951) analyzes qualitative properties of a second-order nonlinear ordinary differential equation (ODE; see Eqn. (4.2)) which arises by eliminating delays in the original DDE.

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[1] It must be mentioned, however, that Frisch introduced random shocks in the investment equation, which prevent oscillations from dying down on their own, and thus the economy from gradually settling down into its equilibrium. This can be viewed as an attempt at employing stochastic modeling in economics.

[2] Not to be confused with another well-known model invented by Goodwin, namely the Goodwin model of the class struggle, which is usually referred to as the Goodwin Model.

The original representation as a DDE is scrutinized by Matsumoto, Merlone and Szidarovszky (2018). They investigate the link between delay values and the existence of a limit cycle in the model. Fractional-order formulations of this DDE (as well as a slightly extended version thereof) are studied by Lin et al. (2020).

The last possible formulation of the Goodwin accelerator model is in the form of a dynamical system of first-order differential equations (see Eqn. (4.5)). A general version of this system is proposed by Lorenz (1987), whereas a particular form, which is studied in this article, appears in Lorenz and Nusse (2002). The integer-order version of this system is analyzed in detail by Li et al. (2011). A numerical analysis of a similar model is conducted by He, Yi and Tang (2016).

3 Methodology

In this section we define some of the most important concepts of fractional calculus and state stability theorems which we apply in the next section. Even though we mention three different definitions of fractional-order derivatives, our qualitative analysis is general and does not hinge upon a particular definition. Nevertheless, the presented numerical method, which is used to simulate the system trajectories, is based upon the Grünwald-Letnikov definition. Since the three definitions are equivalent for a wide range of functions, this numerical method generally provides fairly satisfactory approximations[3]. All the definitions and theorems mentioned here can be studied further in Petras (2011) or Podlubny (1999). Let us first define the *differintegral*:

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha}; & \alpha > 0 \\ 1; & \alpha = 0 \\ \int\limits_a^t (d\tau)^\alpha; & \alpha < 0 \end{cases} \tag{3.1} $$

An important special function used in fractional calculus is the (Euler's*) gamma function* defined as follows:

$$ \Gamma(x) = \int\limits_0^\infty t^{x-1} e^{-t} dt \tag{3.2} $$

[3] Nonetheless, extra vigilance is definitely in place when dealing with chaotic fractional-order systems numerically owing to the fact that fractional-order numerical methods are not as developed as their integer-order counterparts just yet (one ought to treat integer-order chaotic systems with the utmost care as well), and seemingly insignificant changes in initial conditions and/or parameters of chaotic systems may have drastic consequences for the system trajectories. Some issues with numerical simulations of chaotic (or even stiff) systems could be partially avoided by taking an extremely small step size, which would, however, increase the computational complexity considerably. Therefore, numerical simulations presented (not only) in this article should be regarded merely as crude depictions of the system trajectories.

The Grünwald-Letnikov, *Riemann-Liouville* and *Caputo derivatives* are defined in the following fashion, respectively:

$$ {}^{G-L}_{\ a}D^{\alpha}_{t} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^{j} \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)} f(t-jh) \tag{3.3} $$

$$ {}^{R-L}_{\ a}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{a}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau\, ;\ n-1<\alpha<n \tag{3.4} $$

$$ {}^{C}_{a}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau\, ;\ n-1<\alpha<n \tag{3.5} $$

Let us now consider the following fractional-order differential equation:

$$ {}_{a}D^{\alpha}_{t} y(t) = f(y(t),t) \tag{3.6} $$

Based upon the Grünwald-Letnikov definition, we can approximate its solutions numerically:

$$ y(t_k) = f(y(t_k),t_k)h^{\alpha} - \sum_{i=v}^{k} c_i^{(\alpha)} y(t_{k-i}) \tag{3.7} $$

where:

$$ c_0^{(\alpha)} = 1;\ c_i^{(\alpha)} = \left(1 - \frac{1+\alpha}{i}\right) c_{i-1}^{(\alpha)} \tag{3.8} $$

Let us look at a *fractional-order dynamical system* (bold denotes vectors and matrices):

$$ D^{\boldsymbol{\alpha}} \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}) \tag{3.9} $$

We say that $\boldsymbol{x}^*$ is an *equilibrium* (*steady state*) of (3.9) iff[4]:

$$ \boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0} \tag{3.10} $$

Theorem 1: An equilibrium of a commensurate[5] fractional-order nonlinear dynamical system is locally asymptotically stable if all eigenvalues of the Jacobian matrix of the system evaluated at the equilibrium satisfy the following condition (if they lie in the stable region of the complex plane):

$$ \left|\arg\left(eig(\boldsymbol{J})\right)\right| = |\arg(\lambda_i)| > \alpha\frac{\pi}{2} \tag{3.11} $$

Theorem 2: An equilibrium of an incommensurate[6] fractional-order nonlinear dynamical system is locally asymptotically stable if all roots of Equation (3.12) satisfy the condition in Equation (3.13):

$$ det\left(diag\left(\lambda^{lcm(q_1,\ldots,q_n)\alpha_1},\ldots,\lambda^{lcm(q_1,\ldots,q_n)\alpha_n}\right) - \boldsymbol{J}\right) = 0 \tag{3.12} $$

$$ |\arg(\lambda)| > \frac{1}{lcm(q_1,\ldots,q_n)}\frac{\pi}{2} \tag{3.13} $$

where $\alpha_i := \frac{p_i}{q_i}$ is the derivative order of the *ith* equation and lcm denotes the least common multiple.

An equilibrium is called a *saddle point* if at least one eigenvalue lies in the stable region and at least one in the unstable region.

[4] if and only if

[5] Which means that all derivative orders are equal.

[6] Which means that not all derivative orders are equal.

4 Stability Analysis of the Model

Goodwin (1951) proposes the following delay differential equation (DDE) to model the multiplier-accelerator interaction in the economy:

$$\varepsilon\frac{dx(t+\theta)}{dt}+(1-\alpha)x(t+\theta)=O_A(t+\theta)+\varphi\left(\frac{dx(t)}{dt}\right) \tag{4.1}$$

Approximating the delayed terms linearly using their respective Taylor series expansions and shifting $O_A(t+\theta)$ by θ units in time, he arrives at the following second-order nonlinear ordinary differential equation:

$$\varepsilon\frac{d^2x(t)}{dt^2}+(\varepsilon+(1-\alpha)\theta)\frac{dx(t)}{dt}-\varphi\left(\frac{dx(t)}{dt}\right)+(1-\alpha)x(t)=O^*(t) \tag{4.2}$$

It ought to be noted here that Eqn. (4.2) does not necessarily exhibit the same properties as Eqn. (4.1). Therefore, it is advisable that numerical simulations of the solution to the original DDE be carried out to inspect whether dropping higher-order terms in the Taylor series may have caused any significant changes in the qualitative properties of the model. That said, it was not until a few decades after the model had first been published that a vast majority of contemporary numerical methods which are capable of efficiently and effectively approximating solutions to DDEs became widely available. Moreover, advanced analytical techniques which are nowadays utilized for closely examining DDEs had yet to be introduced as well.

Lorenz (1987) generalizes Eqn. (4.2) in the following manner:

$$\frac{d^2x(t)}{dt^2}+A\big(x(t)\big)\frac{dx(t)}{dt}+B\big(x(t)\big)=O^*(t) \tag{4.3}$$

where $A\big(x(t)\big)$ is an even function such that $A(0)<0$, $B\big(x(t)\big)$ is an odd function with $B(0)=0$. We are particularly interested in a specific form of this generalization considered by Lorenz and Nusse (2002):

$$\frac{d^2x(t)}{dt^2}+\alpha\frac{x^2(t)-1}{x^2(t)+1}\frac{dx(t)}{dt}-\omega_0x(t)+\delta x^3(t)=f\sin(\Omega_1t) \tag{4.4}$$

As is shown in Li et al. (2011), Eqn. (4.4) can be rewritten as the following dynamical system:

$$\begin{aligned}\frac{dx(t)}{dt}&=y(t)\\ \frac{dy(t)}{dt}&=-\alpha\frac{x^2(t)-1}{x^2(t)+1}y(t)+\omega_0x(t)-\delta x^3(t)+f\sin(\Omega_1t)\end{aligned} \tag{4.5}$$

In this article we consider a generalized version of Eqn. (4.5):

$$\begin{aligned}{}_0D_t^{q_1}x(t)&=y(t)\\ {}_0D_t^{q_2}y(t)&=-\alpha\frac{x^2(t)-1}{x^2(t)+1}y(t)+\omega_0x(t)-\delta x^3(t)+f\sin(\Omega_1t)\end{aligned} \tag{4.6}$$

where $\alpha\coloneqq\varepsilon\tilde{\alpha}$, $f\coloneqq\varepsilon\tilde{f}$, ω_0, δ and Ω_1 are parameters of the model and $\varepsilon\geq 0$ is a perturbation parameter. Parameters q_1 and q_2 denote derivative orders and are assumed to be positive rational numbers less than 2. Although parameters α and ω_0 are assumed positive, which stems from the aforementioned conditions imposed upon functions

$A(x(t))$ and $B(x(t))$ from Eqn. (4.3), we do not restrict them to being positive in our analysis. But at the same time in order to ensure analytical tractability, we only analyze the steady-state stability of the nonperturbed version of Eqn. (4.6), i.e., we assume $\varepsilon = 0$. The perturbed version, which is described in Eqn. (4.6), is subsequently analyzed numerically.

4.1 Equilibria of the Nonperturbed System

In order to obtain equilibria of the nonperturbed version of Eqn. (4.6), one needs to solve the following nonlinear system of algebraic equations:

$$\begin{aligned} 0 &= y_e \\ 0 &= \omega_0 x_e - \delta x_e^3 \end{aligned} \tag{4.7}$$

In case parameters ω_0 and δ are non-zero with the same sign, there are three distinct equilibria[7], namely $(0,0)$, $(\pm\sqrt{\frac{\omega_0}{\delta}}, 0)$. If either parameter (but not both) is equal to zero, or if the parameters have opposite signs, only one distinct equilibrium exists - $(0,0)$. Should both parameters be zero at the same time, then there would be infinitely many equilibrium points with coordinates $(c,0); c \in \mathbb{R}$. All eigenvalues are then, however, zero so we omit this case in our analysis altogether. The same is true when $\omega_0 = 0$ so we do not analyze this case here, either.

4.2 Steady-State Stability of the Commensurate Nonperturbed System

The first step of the analysis is to evaluate the Jacobian matrix at each steady state. For the nonperturbed system it has the following form:

$$\boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ \omega_0 - 3\delta x_e^2 & 0 \end{pmatrix} \tag{4.8}$$

(1) Let us first investigate $(0,0)$. The characteristic equation for this equilibrium is:

$$\lambda^2 - \omega_0 = 0 \tag{4.9}$$

If $\omega_0 > 0$, then the corresponding eigenvalues are $\lambda_{1,2} = \pm\sqrt{\omega_0}$. Hence, the equilibrium is a saddle point for the derivative orders considered in this article. If $\omega_0 < 0$, the eigenvalues become $\lambda_{1,2} = \pm\sqrt{\omega_0}i$. Hence, if the derivative order is less than 1, the equilibrium is locally asymptotically stable according to Theorem 1. Since $tr(\boldsymbol{J}) = 0$ and $det(\boldsymbol{J}) > 0$, the equilibrium becomes a stable center when the derivative order is 1. For higher orders, both eigenvalues lie in the unstable region.

(2) Since the equilibrium point in the Jacobian matrix is raised to the second power, we can analyze both equilibrium points $(\pm\sqrt{\frac{\omega_0}{\delta}}, 0)$ simultaneously. The characteristic equation in this case becomes:

$$\lambda^2 + 2\omega_0 = 0 \tag{4.10}$$

[7] Here, the abscissa represents x_e and the ordinate represents y_e.

with eigenvalues either $\lambda_{1,2} = \pm\sqrt{-2\omega_0}$ if $\omega_0 < 0$ or $\lambda_{1,2} = \pm\sqrt{-2\omega_0}i$ if $\omega_0 > 0$. In the first case, the equilibria are saddle points. In the second case, the equilibria are asymptotically stable if the derivative order is less than 1. When it is identically one, the equilibria are stable centers. In case of higher orders, the eigenvalues lie in the unstable region so chaotic behavior can potentially occur.

4.3 Steady-State Stability of the Incommensurate Nonperturbed System

(1) Let us start by analyzing $(0, 0)$. The corresponding characteristic equation for the incommensurate system is as follows:

$$\lambda^{M(q_1+q_2)} - \omega_0 = 0 \tag{4.11}$$

where M is the least common multiple[8] discussed in Theorem 2. There are exactly $M(q_1 + q_2)$ complex solutions to this equation, which is a direct corollary of the well-known fundamental theorem of algebra. In order to establish stability conditions, their respective arguments need to be analyzed.

Let us first assume that $\omega_0 > 0$. It follows from the de Moivre's formula that their respective arguments are of the form[9] $\frac{2\pi k}{M(q_1+q_2)}; k = 0, 1, \dots, M(q_1 + q_2) - 1$. One can immediately notice that for $k = 0$, the solution lies in the unstable region for all derivative orders considered in the article. Let us then derive a condition for which the equilibrium is a saddle point. Since one root of the equation is always guaranteed to lie in the unstable region, we now seek to ensure that at least one root be in the stable region. That entails putting a constraint upon $q_1 + q_2$ so that $\frac{2k\pi}{M(q_1+q_2)} > \frac{\pi}{2M}$ and $\frac{2k\pi}{M(q_1+q_2)} < 2\pi - \frac{\pi}{2M}$ for the same k[10]. It can be verified directly using simple algebraic techniques that the *kth* root lies in the stable region if[11] $\frac{4k}{4M-1} < q_1 + q_2 < 4k$.

If $\omega_0 < 0$, the corresponding arguments are of the form $\frac{\pi+2\pi k}{M(q_1+q_2)}; k = 0, 1, \dots, M(q_1 + q_2) - 1$. It can be derived applying the same reasoning as above that the *kth* root lies in the stable region if $\frac{4k+2}{4M-1} < q_1 + q_2 < 4k + 2$. It is locally asymptotically stable if all roots lie in the stable region, which is ensured when $\frac{2}{4M-1} < q_1 + q_2 < 2$.

(2) The characteristic equation for $(\pm\sqrt{\frac{\omega_0}{\delta}}, 0)$ is:

$$\lambda^{M(q_1+q_2)} + 2\omega_0 = 0 \tag{4.12}$$

The constant term of the polynomial is nearly identical to the one in Eqn. (4.11), except for the fact that it has the opposite sign and twice the modulus. Hence, the solutions to

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[8] More precisely, it is a natural number associated with the least common multiple.

[9] Since $M(q_1 + q_2) \geq 2$ and the characteristic polynomial only has the leading coefficient and the constant term, there are always at least 2 distinct roots.

[10] Where $k = 1, \dots, M(q_1 + q_2) - 1$.

[11] Since $M \geq 2$, $\frac{4k}{4M-1} \leq \frac{4k}{7}$. This upper bound is useful for $k = 1$. The other fraction containing M is practical for $k = \mathrm{M}(q_1 + q_2) - \mathrm{n}; \mathrm{n} = 1, \dots, \mathrm{M}(q_1 + q_2) - 2$.

Eqn. (4.11) for $\omega_0 < 0$ have the same arguments as those to Eqn. (4.12) for $\omega_0 > 0$ and vice versa. Therefore, conclusions concerning stability of these two equilibria are essentially the same as in the previous case; the only difference being that they apply to ω_0 with the opposite sign.

5 Numerical Analysis of the Model

In this section, we present numerical simulations of the nonperturbed as well as the perturbed version of the system described in Eqn. (4.6) based upon a numerical technique explained in Section 3. For this purpose, we have programmed a MATLAB function with the help of a toolbox published by Petras (2021). The colors pertain to specific derivative orders: blue to [1, 1], red to [0.95, 0.95], green to [0.9, 0.9] and yellow to [1, 0.9].

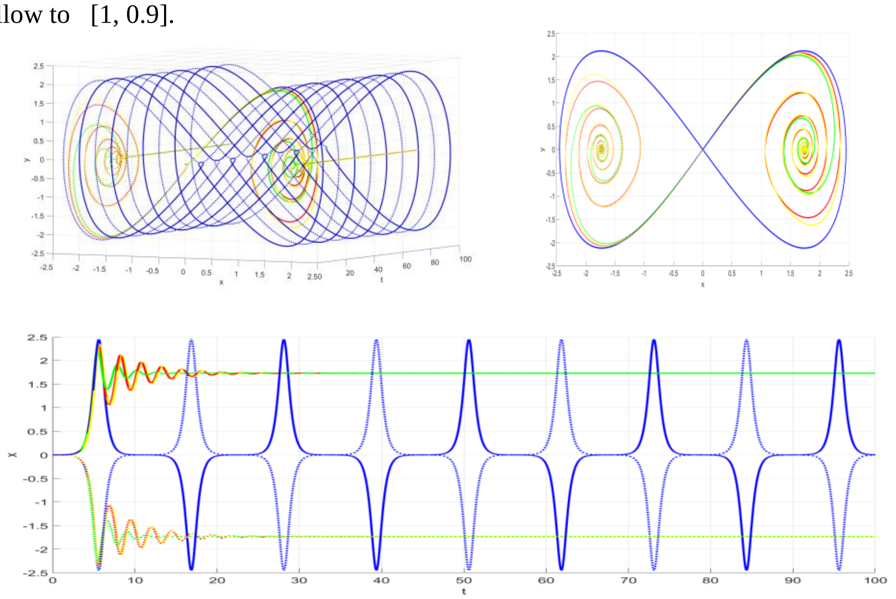

Fig. 1. Numerical simulations of the nonperturbed system with parameter values $\delta = 1$, $\omega_0 = 3$. Solid and dotted lines represent the following initial conditions - [0, 0.001], [0, -0.001]. The left picture depicts selected trajectories in the *x*-*t*-*y* plane while the right one in the *x*-*y* plane. In the bottom picture, the evolution of *x* in time is depicted.

As shown in Fig. 1., incorporation of memory in the model by means of fractional calculus can bring stability to the system. In the integer-order case for the selected parameters and initial conditions, a homoclinic orbit can be observed. The system oscillates around its two non-zero equilibria (which are stable centers), joining the zero equilibrium (which is a saddle point) to itself. If we decrease the derivative orders (so that the conditions for local asymptotic stability derived in the previous section are satisfied), the two non-zero equilibria become stable foci[12] - trajectories starting nearby spiral towards them in time.

Fig. 2. depicts a perturbed version of the system. The system appears to behave chaotically for integer orders. Lowering the derivative orders causes the system trajectories to oscillate around one of the non-zero equilibria in a somewhat regular fashion. Once again, certain fractional orders have turned a rather chaotic system into a relatively stable one.

Similar observations could be made for other initial conditions and parameters as well. Carefully-chosen fractional orders can sometimes bring stability to otherwise chaotic systems.

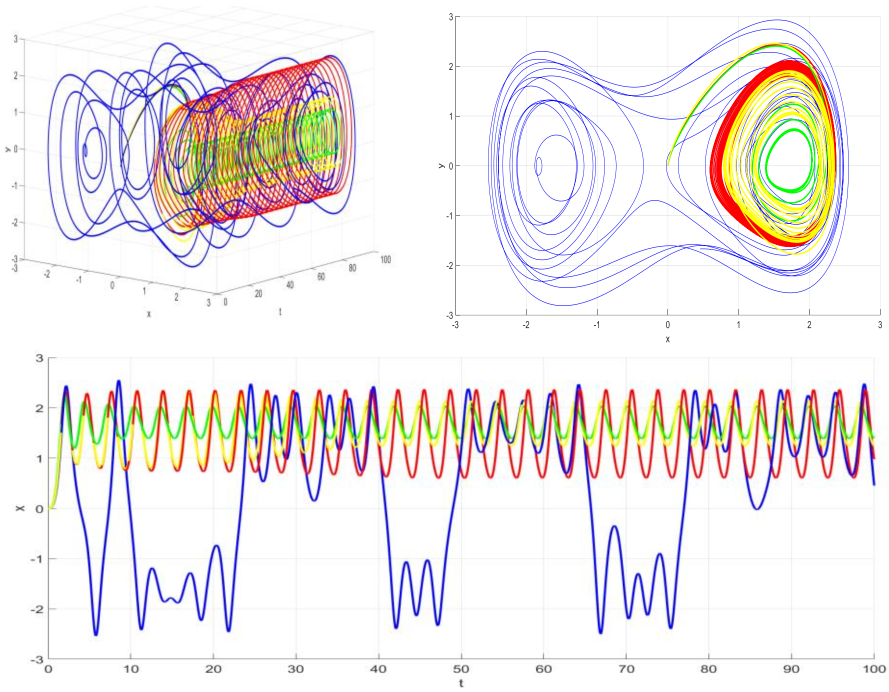

Fig. 2. Numerical simulations of the perturbed system with parameter values $\delta = 1$, $\omega_0 = 3$, $\alpha = 0.5$, $f = 0.95$, $\Omega_1 = 2$ and the initial condition [0, 0.001]. The left picture depicts selected trajectories in the *x*-*t*-*y* plane while the right one in the *x*-*y* plane. In the bottom picture, the evolution of *x* in time is depicted.

[12] Sometimes called focus points or spiral points.

6 Conclusion

As has been shown in this article, fractional calculus can effectively turn unstable or neutrally stable equilibria into locally asymptotically stable ones. This incorporation of memory (or a certain kind of backward-looking expectations if you will) may cause chaotic systems to behave relatively predictably. Not only do nearly all economic processes depend upon their past states and expectations, but there is also another argument[13] to consider, which favors using fractional calculus in economics more extensively. Economic phenomena are usually very complex. Even highly simplified mathematical formulations thereof can lead to chaotic models, in which a subtle change in parameters and/or initial conditions can turn a stable, equilibrium-approaching system into a system which blows up. However, these blow-ups do not reflect what can be observed in the real world whatsoever. Even though economic processes can evolve in an oscillatory behavior and abrupt shifts can be detected therein, rarely (if ever at all) do we see them violently spiral out of their way and never come back, just because a parameter or an initial condition has slightly changed[14]. If this were the case, we might just be lucky to have been born into a world with stable parameters and initial conditions. It is much more likely that our world, however complex it may be, is structurally more stable than its mathematical depiction with integer-order dynamical systems, and the observed oscillations and sudden changes may just be a result of perturbations affecting the system.

This article does not purport to substantiate this claim. It only provides evidence that fractional-order derivatives may have a significant impact upon the steady-state stability of dynamical systems. Therefore, if economists aim to model a system which greatly depends upon its past states, fractional calculus might be an invaluable tool in their mathematical toolkit.

[13] The main idea behind this argument was put forth by Martin Šuster, the director of economic research at the National Bank of Slovakia, when the author of this article was discussing economic modeling with him.

[14] In some cases, one might argue that a small change in a specific parameter can lead to major changes, after all. For instance, the (entirely for some) different paths upon which West and East Germany or South and North Korea embarked after they had split up and different political and economic ideologies had been adopted. The question is, however, whether these changes were actually subtle (as changes can impact system trajectories significantly even in otherwise stable fractional-order systems, as is shown in our analysis as well when ω_0 changes signs). Even if that was indeed the case, both systems despite the change in parameters might still have been converging to the very same mutual equilibrium. The convergence may just have been expedited (or impeded, for that matter).

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