The Consumer Surplus Line Integral Revisited

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https://doi.org/10.53465/EDAMBA.2022.9788022550420.453-461

Abstract. The consumer surplus line integral is a concept which has helped shed some light upon the consumer's welfare changes due to changes in product prices and/or changes in the consumer's income. The main objective of this article is to derive the consumer surplus line integral making use of the divergence theorem as well as Green's theorem. This approach enables the interested reader to come up with other line integrals with the same value. To our knowledge, this is one of the very first direct applications of the two theorems in economics. A partial objective is to summarize the fundamental ideas, definitions and theorems from a branch of vector calculus dealing with curves and line integrals. Therefore, the target audience of the article consists of not only economists studying the consumer's surplus, but also of quantitative-minded economists seeking for new research methods.

Keywords: Consumer's Surplus, Vector Calculus, Line Integrals

JEL classification: *C60, C69, D11*

1 Introduction

Vector calculus plays an important role in natural sciences, namely in physics. Some of the most fundamental laws of nature, such as those of electromagnetism described by Maxwell's equations, are formulated making use of the notions like the gradient, divergence, curl and the Laplacian or the line, surface and volume integral as well as theorems such as the divergence theorem, Green's and Stokes' theorem. Yet its direct use in economics is somewhat meager. This comes as a surprise since most branches of mathematical analysis alone, including fractional, stochastic and time-scale calculus, or ordinary and partial differential equations, have successfully been employed in economics. An even better reason why one could expect the use of vector calculus in economics is the fact that vectors and matrices show up quite often in economic models. For instance, the Leontief input-output analysis could benefit from a direct application of vector calculus.

Notwithstanding the above, there is one line of economic research which utilizes some of the aforementioned techniques to analyze changes in the consumer's surplus and his/her utility. The study was pioneered by [6]. A line integral is derived which can be used (under certain assumptions) to calculate the change in the consumer's surplus should a price (or a set of prices for multiple products) change. Even though the integral is derived in a straightforward manner, it can also be obtained by an application of Green's theorem, which also provides a wider range of admissible integrands for the line integral.

Therefore, the objective of this article is to derive the consumer surplus integral making use of Green's theorem, which would be one of the very first direct applications of the theorem in economics. A partial objective is to summarize the main ideas, definitions, and theorems of a part of vector calculus analyzing curves and line integrals. The target audience of the article consists of economists working with the consumer surplus integral as well as quantitative-minded economists seeking for new methods. In Section 2 we provide the reader with the review of pertinent literature. Section 3 provides some of the fundamental definitions, theorems and ideas of vector calculus. In Section 4, the consumer surplus integral is derived, and the results are discussed in Section 5.

2 Literature Review

As has already been mentioned, the concept of the consumer's surplus can be traced back to [6]. The concept alongside with the "Marshallian triangles" was later popularized by [9]. A major pertinent contribution was made by [7] and his notion of compensating variations. Some of the very first formulations of the consumer line integral can be found in [8] and [12].

Since the discussed integral is of a vector field (or in other words, it is orientable and can, therefore, depend upon the integration path), it is important to study the assumptions under which it is path independent. This is in part done in [14]. A concise treatment of the issue can be found in [3]. The authors make a summary of the fundamental theory of path independence for line integrals. Some of the other properties of the consumer surplus line integral are studied in [4].

The consumer surplus line integral has found numerous applications in the discrete choice theory. For instance, [10] introduces the concept of random compensating variation and proves its equivalence with a line integral. The results of the paper are analyzed and applied even more, for instance in [5].

3 Curves, Line Integrals and Green's Theorem

In this section, we provide an overview of the basic concepts of vector calculus. A rigorous mathematical treatment thereof can be found in [1], [2] and [11]. A less rigorous approach is taken in [13].

Definition 1. Let $n \in \mathbb{N}, n > 1$ and $\phi: \mathbb{R} \supset [a, b] \longrightarrow \mathbb{R}^n, t \longmapsto \phi(t) = \big(\phi_1(t), \dots, \phi_n(t)\big), \phi \in C([a, b], \mathbb{R}^n)$. Then the set $\mathcal{K} \coloneqq \phi([a, b]) = \{x \in \mathbb{R}^n : \exists t \in [a, b] : x = \phi(t)\}$ is called a curve in $\mathbb{R}^n$, the function ϕ its parametrization and t a parameter.

In simple terms, a curve is a continuous image of a compact interval. An economic example of a curve is a price-consumption curve, which can be parametrized with the price of a given good.

An exotic example of a curve is the Hilbert space-filling curve. Owing to this example, some more properties need to be studied before we delve into line integrals.

Definition 2. $\mathcal{K}$ is called a regular curve if $\phi \in C^1([a, b], \mathbb{R}^n)$ and its derivative never vanishes.

This definition simply states that a regular curve is a smooth curve without spikes (if the curve described the position of a fly in time, this would imply that the fly cannot change the flight direction discontinuously).

Definition 3. $\mathcal{K}$ is called a Jordan curve if at least one of its parametrizations ϕ is bijective. Moreover, $\mathcal{K}$ is called a closed Jordan curve if $\phi(a) = \phi(a)$ and ϕ restricted to (a, b) is bijective.

A Jordan curve is such a curve which does not cross itself. From here on, we require that every curve we analyze be a piecewise regular (meaning it can be partitioned into regular curves) Jordan curve or a piecewise regular closed Jordan curve.

Theorem 4. If $\phi \in C^1([a, b], \mathbb{R}^n)$ is a Jordan parametrization of a Jordan curve $\mathcal{K}$. Then the length of the curve can be computed as follows:

$$\mathcal{L}(\mathcal{K}) = \int_a^b \|\phi'(t)\| \, dt \tag{1}$$

Sketch of the proof. We can approximate the length of the curve by summing up lengths of line segments connecting points on the curve:

$$\mathcal{L}(\mathcal{K}) \approx \sum_{i=1}^{n} \|\phi(t_i) - \phi(t_{i-1})\| \tag{2}$$

Since ϕ is assumed continuously differentiable, we can apply the mean value theorem to obtain:

$$\mathcal{L}(\mathcal{K}) \approx \sum_{i=1}^{n} \|\phi'(c_i)\| \, \Delta t_i, c_i \in (t_{i-1}, t_i) \tag{3}$$

Taking the limit, we get the desired result.

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It ought to be noted here that we have just sketched an informal proof of a weaker theorem than Theorem 4 which states how to compute the length of a parametrization, not the curve. As it turns out, however, the length of a Jordan curve is independent of its Jordan parametrization.

The following concept of a function of the curve length (also referred to as arc length) is of great importance when studying line integrals.

Definition 5. Let $\phi \in C^1([a,b], \mathbb{R}^n)$ be a Jordan parametrization of a Jordan curve $\mathcal{K}$. Then we define a function s: $[a,b] \longrightarrow \mathbb{R}, t \mapsto \int_a^t \|\phi'(\tau)\| \, d\tau$.

For every admissible t, the function $s(t)$ measures the length of the Jordan curve $\mathcal{K}$ up to t. Since the integrand is a continuous function, we can find the differential of s in the form: $ds = s'(t)dt = \|\phi'(t)\|dt$. Also note that if $\mathcal{K}$ is of finite length on a bounded interval, then s is of bounded variation on the interval. We are now ready to define the line integral of scalar as well as vector-valued functions.

Definition 6. (Line integral of a scalar function[1]) Let $\mathcal{K} = \phi([a,b]) \subset \mathbb{R}^n$ be a Jordan curve of finite length with its regular Jordan parametrization ϕ. Let $f\colon \mathbb{R}^n \supseteq D(f) \longrightarrow \mathbb{R}$ such that $\mathcal{K} \subset D(f)$. Then a line integral of the scalar function f along the curve $\mathcal{K}$ is defined as follows:

$$\int_{\mathcal{K}} f(x)ds \coloneqq \int_a^b f\big(\phi(t)\big)\, s'(t)dt = \int_a^b f\big(\phi(t)\big)\|\phi'(t)\|dt \tag{4}$$

If $\mathcal{K}$ is closed, the following notation is sometimes used:

$$\oint_{\mathcal{K}} f(x)ds \tag{5}$$

As can be seen, the line integral is defined as a Riemann-Stieltjes integral with respect to s (which in this case is of bounded variation, as already noted). Geometrically, it

[1] Sometimes referred to as a line integral of the first kind.

measures the area between a curve $\mathcal{K}$ and the graph of a function f defined along the curve.

Definition 7. (Line integral of a vector-valued function[2]) Let $\mathcal{K} = \phi([a,b]) \subset \mathbb{R}^n$ be a Jordan curve of finite length with its regular Jordan parametrization ϕ. Let $F: \mathbb{R}^n \supseteq D(F) \longrightarrow \mathbb{R}^n, x \longmapsto F(x) = \big(f_1(x), \dots, f_n(x)\big)$ such that $\mathcal{K} \subset D(F)$. Then a line integral of the vector-valued function F along the curve $\mathcal{K}$ with its parametrization ϕ is defined as follows:

$$\int_{\phi} F(x) \cdot dx = \int_{a}^{b} F\big(\phi(t)\big) \cdot \phi'(t)dt = \sum_{i=1}^{n} \int_{a}^{b} f_i\big(\phi(t)\big)\, \phi'_i(t)dt = \int_{\phi} \sum_{i=1}^{n} f_i(x)dx_i \quad (6)$$

If $\mathcal{K}$ is closed, the following notation is sometimes used:

$$\oint_{\phi} F(x) \cdot dx \tag{7}$$

The notation indicates that this line integral is in general not independent of the parametrization (it can be shown that it gives the same result for equivalent regular Jordan parametrizations, but different orientations impact the sign). Later in the next, however, we write $\int_{\mathcal{K}} F(x) \cdot dx$ when the orientation is given.

As the following sequence of steps indicates, there is a relation between the line integrals of the first and the second kind:

$$\int_{\phi} F(x) \cdot dx = \sum_{i=1}^{n} \int_{a}^{b} f_i\big(\phi(t)\big)\, \phi'_i(t)dt = \int_{a}^{b} \left(\sum_{i=1}^{n} f_i\big(\phi(t)\big) \frac{\phi'_i(t)}{\|\phi'(t)\|} \right) \|\phi'(t)\|dt =$$
$$= \int_{\mathcal{K}} F(x) \cdot \frac{\phi'(t)}{\|\phi'(t)\|} ds \tag{8}$$

At the very end of this section, let us formulate two important results of vector analysis, namely the divergence theorem and its consequence Green's theorem, which will be used in the next section.

Theorem 8. (The divergence theorem in $\mathbb{R}^2$) Let $B \subset \mathbb{R}^2$ be a compact set with a piecewise smooth boundary ∂B. Let $F(x,y) = \big(f_1(x,y), f_2(x,y)\big)$ such that $f_i \in C^1(B, \mathbb{R}), i = 1, 2$. Then

[2] Sometimes referred to as a line integral of the second kind.

$$\iint_B \nabla \cdot F \, dxdy = \oint_{\partial B} F \cdot \upsilon \, ds \tag{9}$$

where υ is the outward unit normal vector (the path is oriented anticlockwise) and $\nabla \cdot F := \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$ is the divergence of a vector-valued function.

Theorem 9. (Green's theorem) Let $B \subset \mathbb{R}^2$ and $f_1(x,y), f_2(x,y)$ have the same properties as in the previous theorem. Then for any anticlockwise-oriented parametrization ϕ of ∂B

$$\iint_B \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy = \oint_\phi f_1 \, dx + f_2 dy \tag{10}$$

4 Derivation of the Consumer Surplus Integral

In this section, we derive the consumer surplus line integral making use of Green's theorem. Let us consider a continuously differentiable demand function $f: \mathbb{R}_+ \longrightarrow \mathbb{R}_+, p \longmapsto f(p) = q$. Let there be an increase in the price from p_1 to p_2. The problem is to calculate the change in the consumer's surplus denoted by T. It is quite evident that the change is the shaded area in Figure 1.

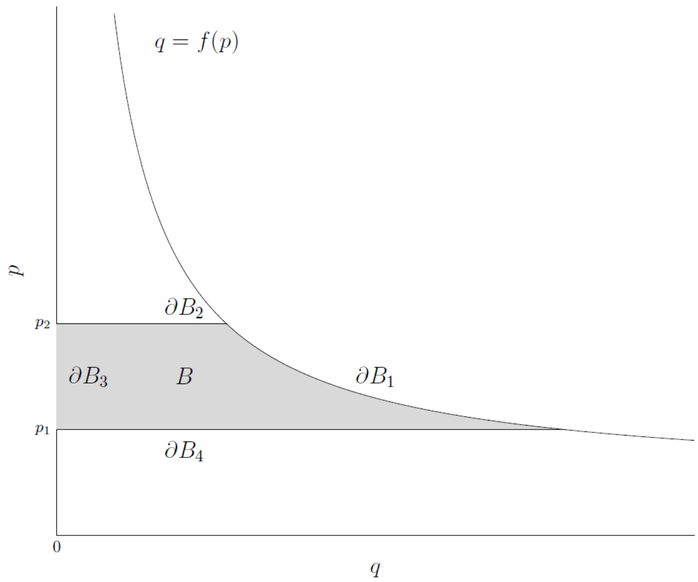

Figure 1 The Consumer Surplus Line Integral
Source: Own illustration

It can immediately be noticed that the area can be calculated as the following integral:

$$T = -\int_{p_1}^{p_2} f(p)\, dp \tag{11}$$

This integral could be thought of as a line integral along the price-consumption curve, which could be parametrized with the price. The change in the consumer's surplus can also be calculated as the following double integral:

$$T = -\iint_B 1 dq dp \tag{12}$$

where $B = \{(q,p) \in \mathbb{R}^2 : p_1 \le p \le p_2 \wedge 0 \le q \le f(p)\}$. Making use of Fubini's theorem, we can rewrite the double integral as an iterated integral:

$$T = -\int_{p_1}^{p_2} \left(\int_0^{f(p)} 1 dq \right) dp \tag{13}$$

Evaluating the inner integral would yield Equation (11). Observe that the set B with its boundary ∂B satisfies the assumptions of Green's theorem. Therefore, we can look for a continuously differentiable vector-valued function $F(q,p) = \big(f_1(q,p), f_2(q,p)\big)$ such that $\frac{\partial f_2}{\partial q} - \frac{\partial f_1}{\partial p} = 1$. One of the first candidates which come up is the following vector-valued function: $F = (0, q)$. According to Green's theorem, we obtain:

$$T = -\iint_B 1 dq dp = -\oint_{\partial B} 0 dq + q dp \tag{14}$$

As can be seen in Figure 1, the boundary can be split into four parts in the following manner:

$$\partial B = \bigcup_{i=1}^{4} \partial B_i \tag{15}$$

These sets are pairwise disjoint except for a finite number of points. Therefore, thanks to the additivity of the line integral, we can write:

$$-\oint_{\partial B} 0 dq + q dp = -\sum_{i=1}^{4} \int_{\partial B_i} q dp \tag{16}$$

Let us notice that the only potentially non-zero integral of all four integrals is along ∂B_1. Along $\partial B_3, q$ is identically zero, so the whole integral is zero. Along ∂B_2 as well as ∂B_4, q is no longer zero, but the price does not change along these lines, therefore, $dp = 0$. We can parametrize ∂B_1 quite naturally since it is a graph of a function. We let $p = p \in [p_1, p_2]$ and $q = f(p)$. The differentials are as follows: $dp = dp$ and $dq = f'(p)dp$. Hence, we get the desired result:

$$T = -\int_{p_1}^{p_2} f(p)dp \tag{17}$$

This integral is then extended in the literature for the case when multiple prices change. Under certain assumptions, the resulting line integral does not depend upon the integration path and can therefore be evaluated as a line integral along line segments. These assumptions are quite restrictive (see [14]) or [3]).

Using our approach which relies upon Green's theorem, we could derive other line integrals the value of which is equal to that of the integral given by Equation (17) just by selecting a suitable vector-valued function and/or a different anticlockwise regular Jordan parametrization.

One could also use the divergence theorem to derive the integral (which should not come as a surprise since Green's theorem is its consequence). In that case, one might consider $F = (q, 0)$. The outer unit normal vectors for ∂B_2, ∂B_3 and ∂B_4 are $(0 \quad 1)$, $(-1 \quad 0)$ and $(0 \quad -1)$, respectively. Therefore, the only potentially non-zero integral is once again the one along ∂B_1. The outward unit normal vector there (considering the same parametrization used in Green's theorem) is equal to $\frac{1}{\sqrt{(dp)^2+(f'(p)dp)^2}}(dp \quad -f'(p)dp)$ and $ds = \sqrt{(dp)^2 + (f'(p)dp)^2}$, so in the end we get the same integral.

5 Conclusion

In this article we have derived the consumer surplus line integral making use of Green's theorem as well as the divergence theorem. This approach enables readers to come up with numerous line integrals with the same value by selecting a suitable vector-valued function and a parametrization.

As far as we know, this is one of the very first direct applications of the two theorems from vector calculus in economics. In our opinion, vector analysis has the potential to be successfully employed in economics just like it has been employed in physics for nearly two centuries. Vector-valued functions, albeit rarely used in economics, can describe many real-world economic processes. One might consider, for instance, a vector-valued function the inputs of which are factors of production, and the output is a vector with elements equal to the production of industries in the economy. One step further from this example is the reformulation of the Leontief input-output model. The curves in this setting could be curves joining points in the production space corresponding to different prices of the factors of production.

Another possible area of research where vector calculus might prove beneficial is regional economics. Vector-valued functions in this case could assign to each point in space (which might represent a country, a city, or even more abstract structures) a vector of pertinent economic indicators.

There are more examples as to how vector calculus could help economists study and tackle real-world economic phenomena. However, delving into more detail is way beyond the scope of this contribution (or any single article).

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